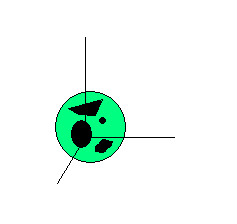
**EM fields asymptotic expansion**

**Radiation emitted by more general charge distributions**

Suppose we have a charge/current distribution inside a ball, changing with time.



We’d like to calculate the EM field at distances far from the inside of the ball. Well, we can calculate φ, and **A** from above. But we need not calculate both if all we want is EM *outside* the ball. We can simply calculate **A** from above. And then get **B**. And then we can use ME to obtain **E,** exploiting the internal consistency . For outside the distribution we have,



and (4) tells us that,



So we have,



**Fourier decomposition of charge/current distribution**

Suppose we have a bunch of charges running around in a sphere. This will have consequent charge density that changes with time, and current density. We’ll assume that these functions are periodic in time. If they aren’t harmonic, then we can do a Fourier transform on them to convert them to the required form. For instance, we’d have:



Note that since ρ(**r**,t) and j(**r**,t) are real functions, it must be the case that:



and so,



And similarly for **j**. Now when we calculate **A** we will get terms like this below.



And similarly for **B** and **E** when we take the curl and curl + integral respectively.

Then we will want to calculate **S** from this. **S** is:



The power will fluctuate with time if the charges/currents are fluctuating, and we don’t really want to know the time-dependence. We will want to look at the time-averaged power, that is the power averaged over a period. And this will be.



Now all the terms in this summand will average to zero due to the oscillations – unless it happens that ωm = -ωn and the oscillations cancel out. So picking out these terms we have:



and so,



The only ‘problem’ with this is that a sinusoid will have two such Fourier components, and the product 4 (only 2 of which will be nonzero) and this is sort of cumbersome. So on the other hand, if we represent (usually by inspection) ρ and **j** as…



– we could use the imaginary part instead of course, then **A**, **E**, and **B** will come out similarly.



And forming the power.



Now taking a time average we get,



Keeping in mind that ωm and ωn must be greater than or equal to 0, only the 2nd and 3rd terms will be non-zero. So we’ll have,



And so,



I think we’ll usually do it this way. So…

**Expansion of A in powers of r and collecting just the 1/r ‘radiation’ terms**

We’d like to calculate the EM field and the radiation being emitted outside the sphere containing the charges. Just as outlined above, we calculate **A** via:



First we plug in **j**(r,t), and get



Now we usually do a large **r** expansion/approximation. Use



Then we can plug this approximation into the exponent and denominator. Note we don’t need to put in higher order terms in √ expansion because they will bring down factors of 1/r which go away in the large r limit. For the same reason there is no need to expand the denominator |**r**-**r**′| because that will just bring down more powers of 1/r. So only the Taylor series expansion of the exponent about r′ = 0 will be necessary. So filling this approximation into the exponent and Taylor expanding we get the following:



Note that this term takes the form of a spherical wave with some to-be-determined coefficient.

**Relation of terms in series to multi-pole moments of current/charge distribution**

The zeroth order term will give us the electric dipole contribution to the radiation. The first order term will give us the magnetic dipole and electric quadrupole contribution. Let’s look at the zeroth order term.



Now we have to calculate the integral over the current density and relate it to the electric dipole moment. Note the integral isn’t zero in this case because we’re not dealing with the steady state – divergence-less current we dealt with in the statics case. We can relate the integral to the electrical dipole moment. Consider:



and so then we can write,



Since **j** is harmonic, **P** will be too. So we’ll be able to write: **P**(t) = **P**e-iωt. Then we have,



Now let’s look at the next term in the series. Now let’s go back to our expression for **A** and look at the next order term – the magnetic dipole part, and electric quadrupole part.



To relate this to the dipole term (and electric quadrupole term we’ll make the following manipulations…



Let’s work on each term separately. The first is:



where we’ve used IBP with the identity (fgh)′ = f′gh + fg′h + fgh′. The boundary term goes to 0 if we stretch the integral out enough to where **j** = 0. Now we can write,



So now we see that the last two terms cancel. Using the continuity equation, we can equate . So we can say:



Now let’s take note that what we really want from **A** is . So we can add to **A** and radial function  since its curl would be 0. We’ll add the function which will make this look like the electric quadrupole moment. So we add:



The second term is simpler:



So putting it all together we have:



So putting it all together, to first order we have:



**Power radiated by electric dipole**

Suppose we have a bunch of charge oscillating back and forth. If the dipole moment of this charge distribution is non-vanishing, then it will constitute the largest term in the expansion. The electric dipole approximation to **A** is found by just keeping the first order term **P**. Once we get **A**, we obtain **B**. Our expression only depends on r and so we can just use the radial part of the gradient. Even if did depend on θ, φ, we would still just use this approximation because the other terms in the spherical coordinate representation of ∇ contain 1/r terms, which would contribute nothing to the radiation in the large r limit.



and then **E**.



Note these formulas seem to say that the electric field oscillates in phase with the dipole moment. So when P points up, E points up, etc. This is in the far-field region of course. Now we can get **S**.



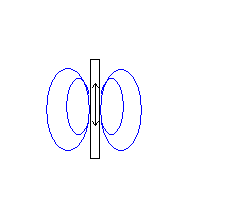
Substituting back in k = ω/c, and μ0 = 1/c2ε0, we can get the power distribution as before in previous file,



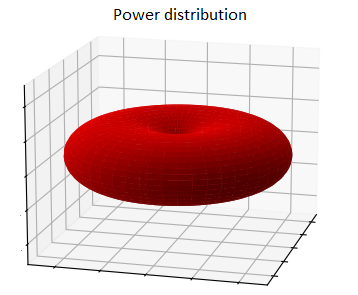
The results are:



where θ is measured from the vertical as usual, along the polarization axis. The power distribution would look something like,



Okay, better picture,



So power is radiated mainly out towards the sides perpendicular to the axis of the dipole, and not at all along the axis. The total power radiated would be:



So we have:



This is the same as the Larmour formula, incidentally.

**Power radiated by magnetic dipole**

Suppose we have a bunch of current sloshing back and forth along an antenna. In that case, the charge distribution is neutral everywhere, so **P** would be 0. Or can think of a current loop magnetic dipole – or more realistically an atomic magnetic dipole – rotating about an axis through the center of the plane of the dipole. **P** would also be zero for this. But **μ** would not be, and so the first order term in the radiation expansion would be the one involving **μ**. So, **A** is approximately,



Note that this is the term we had in the static case as well – with different prefactors. And then for **B** we have,



and for **E**,



And now we need to take the cross-product of the two. This is…



So we have,



And so the power distribution is, orienting our z-axis along **μ**.



and so we have,



and if we integrate over all angles we get,



This is the magnetic equivalent to the Larmour formula.

**Example**

What is the power radiated by a charge circling around a Hydrogen nucleus. Suppose it circles with radius R and frequency ω.

Well, there would be a time-dependent dipole moment:



It also has a magnetic dipole moment:



So electric and magnetic dipole contribution to the radiation is:

